Concepts of Four (World) vector

We have seen that

* $x^2+y^2+z^2$ is variant (not invariant) under Lorentz transformation i.e, $x'^2+y'^2+z'^2 \neq x^2+y^2+z^2$

* $x^2+y^2+z^2-c^2t^2$ is invariant (not variant) under Lorentz transformation i.e, $x'^2+y'^2+z'^2-c^2t'^2 = x^2+y^2+z^2-c^2t^2$

* Laplacian operator ∇^2 i.e, $\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$ is variant (not invariant) under Lorentz transformation i.e, $\nabla'^2 \neq \nabla^2$

  or $\frac{\partial^2}{\partial x'^2}+\frac{\partial^2}{\partial y'^2}+\frac{\partial^2}{\partial z'^2} \neq \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$.

* D'Alembertian operator $\Box^2$ i.e, $\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}$ is invariant (not variant) under Lorentz transformation i.e, $\Box'^2 = \Box^2$ or $\frac{\partial^2}{\partial x'^2}+\frac{\partial^2}{\partial y'^2}+\frac{\partial^2}{\partial z'^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t'^2} = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}$.

Now for understanding the concept of Four vector, we take

$$x^2+y^2+z^2-c^2t^2 = x^2+y^2+z^2+(ict)^2 \quad \because i^2c^2t^2 = -c^2t^2$$

In 4D space, the fourth coordinate may be taken as ict.

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Radius (or position) vector in 4D space:-

Components of radius (or position) vector in 4D space are taken as x_1, x_2, x_3 and x_4 where $x_1 = x$, $x_2 = y$, $x_3 = z$ and $x_4 = ict$.

Specification of any event occuring at a point p in 4-D space may be written as $p(x_1, x_2, x_3, x_4)$. $\cdot p(x_1, x_2, x_3, x_4)$

* Lorentz Transformation equation:

$x' = \gamma(x - vt)$, $y' = y$, $z' = z$ and $t' = \gamma\left(t - \frac{v}{c^2}x\right)$

These Lorentz transformation in Four vector will be understood as follows.

$x' = \gamma(x - vt) \longrightarrow x_1' = \gamma\left(x_1 - v\frac{x_4}{ic}\right)$ $\because x_4 = ict \Rightarrow t = \frac{x_4}{ic}$

$= \gamma\left(x_1 + \frac{i^2 v}{ic} x_4\right)$ $\because i^2 = -1$

$x_1' = \gamma(x_1 + i\beta x_4)$ —— (a) $\because \beta = \frac{v}{c}$

$y' = y \longrightarrow x_2' = x_2$ —— (b)

$z' = z \longrightarrow x_3' = x_3$ —— (c)

and $t' = \gamma\left(t - \frac{v}{c^2}x\right) \longrightarrow ict' = \gamma\left(ict - ic\frac{v}{c^2}x\right) \Rightarrow ict' = \gamma\left(ict - i\frac{v}{c}x\right)$

$\Rightarrow x_4' = \gamma(x_4 - i\beta x_1)$ —— (d) $\because ict = x_4$, $\frac{v}{c} = \beta$

Thus Lorentz transformation of position (radius) four vector will be

$$x_1' = \gamma(x_1 + i\beta x_4), \quad x_2' = x_2, \quad x_3' = x_3 \text{ \& } x_4' = \gamma(x_4 - i\beta x_1) \quad \text{—— (1)}$$

Equⁿ (1) may be written as

$$\left.\begin{aligned} x_1' &= \gamma \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 + i\beta\gamma \cdot x_4 \\ x_2' &= 0 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 \\ x_3' &= 0 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 + 0 \cdot x_4 \\ \text{and } x_4' &= -i\beta\gamma \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 + \gamma \cdot x_4 \end{aligned}\right\} \text{—— (2)}$$

Therefore eqⁿ (1) may be written in matrix form as

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \times \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \quad \text{——— (3)}$$

Eqⁿ (1) may be written in single equation as

$$x_\mu' = \alpha_{\mu\nu} x_\nu \quad \text{——— (A)}$$

where $\mu = 1, 2, 3 \,\&\, 4$, $\nu = 1, 2, 3 \,\&\, 4$.

and $\alpha_{\mu\nu}$ is known as transformation matrix and it is defined as

$$\alpha_{\mu\nu} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \quad \text{——— (B)}$$

or
$$\alpha_{\mu\nu} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} \end{pmatrix}$$

In general, $\boxed{x_\mu' = \alpha_{\mu\nu} \cdot x_\nu}$ ——— (A)

where x_μ' = components of radius (position) four vector in frame S'.

and x_ν = " " " " S.

In general form, $A_\mu' \to$ components of A in frame S'

and $A_\nu \to$ " " " " S

A is said to be four vector if $\boxed{A_\mu' = \alpha_{\mu\nu} \cdot A_\nu}$ ——— (C)

* All four vectors are tensors of rank-1.

Transformation equation of second order (rank) tensor $T_{\mu\nu}$ is given by

$$T'_{\mu\nu} = \alpha_{\mu\lambda} \cdot \alpha_{\nu\sigma} \cdot T_{\lambda\sigma} \quad \text{--- (D)}$$

where $T'_{\mu\nu}$ = 16 components of second rank tensor T in frame s'.

and $T_{\lambda\sigma}$ = 16 components of second rank tensor T in frame s.

$\alpha_{\mu\lambda}$ and $\alpha_{\nu\sigma}$ are transformation matrix (4×4 matrices).

Four vectors with Examples

$$x_\mu = (x_1, x_2, x_3, x_4)$$

or $x_\mu = (\vec{r}, ict)$

where $\vec{r} = x_1\hat{i} + x_2\hat{j} + x_3\hat{k}$ and $x_4 = ict$.

In general, $A_\mu = (\vec{A}, A_4)$ represents four vectors

where $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$ in cartesian coordinates

and A_4 is the fourth component of the four vector A_μ.

Example:- Four wave vector $K_\mu = (K_1, K_2, K_3, K_4)$

$$K_\mu = \left(\vec{K}, \frac{i\omega}{c}\right)$$

where $\vec{K} = K_1\hat{i} + K_2\hat{j} + K_3\hat{k}$ and $K_4 = \frac{i\omega}{c}$.

Ques:- Find the components of four wave vector and show that its squares is zero.

Ans:- phase of a wave vector is defined as $\vec{K}\cdot\vec{r} - \omega t$

and phase of wave vector is relativistic invariant.

i.e, $\vec{K}\cdot\vec{r} - \omega t$ = constant.

where $\vec{K} = K_1\hat{i} + K_2\hat{j} + K_3\hat{k}$ and $\vec{r} = x_1\hat{i} + x_2\hat{j} + x_3\hat{k}$

$\Rightarrow (K_1\hat{i} + K_2\hat{j} + K_3\hat{k})\cdot(x_1\hat{i} + x_2\hat{j} + x_3\hat{k}) + \frac{i\omega}{c}\cdot ict$ = constant $\quad \because i^2 = -1$

$\Rightarrow K_1 x_1 + K_2 x_2 + K_3 x_3 + K_4 \cdot x_4 = \text{constant}$ ——— 1

where $x_4 = ict$ and $K_4 = \frac{i\omega}{c}$ = 4th component of 4-wave vector.

In compact form, equⁿ ① may be written as

$$K_\mu \cdot x_\mu = \text{constant} \quad \text{——— ②}$$

Since x_μ is position 4 vector and dot product of two vectors is scalar and here $K_\mu \cdot x_\mu = \text{scalar}$ so K_μ will be vector.

Thus K_μ will be four vector.

so $K_\mu = (\vec{K}, \frac{i\omega}{c})$

where $\vec{K} = K_1\hat{i} + K_2\hat{j} + K_3\hat{k}$ and $K_4 = \frac{i\omega}{c}$.

Now we have to prove $K_\mu^2 = 0$

Now $K_\mu^2 = K_\mu \cdot K_\mu = K_1^2 + K_2^2 + K_3^2 + K_4^2$

$= K^2 + \left(\frac{i\omega}{c}\right)^2 \qquad \because K^2 = K_1^2 + K_2^2 + K_3^2$ and $K_4 = \frac{i\omega}{c}$

$= K^2 - \frac{\omega^2}{c^2} \qquad \because i^2 = -1$

$= K^2 - K^2 \qquad \because K = \frac{\omega}{c}$ = propagation constant

$\boxed{K_\mu^2 = 0}$ proved.